IIT-JEE 2006 Solutions by MOMENTUM

(questions based on memory of students)

MATHEMATICS

INSTRUCTIONS:

(i) **Question no. 1 to 12 has only one correct option. You will be awarded 3 marks for right answer and –1 mark for wrong answer.**

(ii) **Question no. 13 to 20 has one or more than one correct option(s). You will be awarded 5 marks if you answer all correct options and only correct option(s) and –1 mark will be awarded for wrong answer.**

(iii) **Question no. 21 to 32 are based on small write up first go through it then answer these questions. You will be awarded 5 marks for right answer and –2 marks for wrong answer.**

(iv) **Question no. 33 to 36 are subjective problems. Circle their correct answers. There is no negative marking for it. Each question carries 6 marks.**

(v) **Question no. 37 to 40 carry 6 marks each. These may have more than one correct options. There is no negative marking for these.**

Only one choice is correct

1. $\lim_{x\to 0}\left((\sin x)^{1/x} + \left(\frac{1}{x}\right)^{\sin x}\right)$ **(given** $x > 0$ **)s**

(a) 1 **(b) 0** **(c) –1** **(d) 2**

Sol. **a**

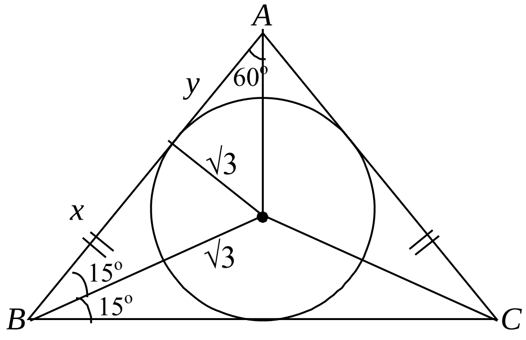
$0 + \infty^o = 0 + 1$ **(since** $x > 0$, so ∞^o is a determinant form**)**

2. *ABC* is a isosceles triangle in which included angle between two equal sides is 120^o and radius of incircle is $\sqrt{3}$, then area of triangle *ABC* is

(a) $12 - 7\sqrt{3}$ sq. units (b) $8 + 7\sqrt{3}$ sq. units

(c) $12 + 7\sqrt{3}$ sq. units (d) $8 - 7\sqrt{3}$ sq. units

Sol. **c**



$$\tan 15^o = \frac{\sqrt{3}}{x}$$

$$x = \frac{\sqrt{3}}{2-\sqrt{3}}$$

$$\tan 60^o = \frac{\sqrt{3}}{y}$$

***y* = 1**

$$x + y = \frac{\sqrt{3}}{2-\sqrt{3}} + 1 = \left(\frac{2}{2-\sqrt{3}}\right)$$

$$\Delta = \frac{1}{2}\times\left(\frac{2}{2-\sqrt{3}}\right)^2 \times \sin 120^o = \left(\frac{\sqrt{3}}{7-4\sqrt{3}}\right)\times\frac{7+4\sqrt{3}}{7+4\sqrt{3}}$$ **= 12 + 7$\sqrt{3}$ sq. unit.**

3. $w = \alpha + i\beta$ is a complex number. The set of z, $z \neq 1$ such that $\frac{w - \bar{w}z}{1-z}$ is purely real number, is

(a) $\{z : z \neq 1\}$ (b) $\{z : |z| \neq 1\}$

(c) $\{z : |z| = 1, z \neq 1\}$ (d) $z = \bar{z}$

Sol. **c**

$$\frac{w-\overline{w}z}{1-z}=\frac{\overline{w}-w\overline{z}}{1-\overline{z}}$$

$\Rightarrow\ w(1-\overline{z})-\overline{w}z(1-\overline{z})=(\overline{w}-w\overline{z})(1-z)\ \Rightarrow\ (w-\overline{w})(|z|^2-1)=0\ \Rightarrow$ **$|z|$ = 1.**

4. If $0<\theta<2\pi$ and $2\sin^2\theta-5\sin\theta+2>0$ then θ lies in the interval

(a) $\left(0,\frac{\pi}{6}\right)\cup\left(\frac{5\pi}{6},2\pi\right)$ (b) $\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$ (c) $(0,\pi)$ (d) $\left(\frac{\pi}{2},2\pi\right)$

Sol. **a**

$$2\sin^2\theta-5\sin\theta+2>0$$

$\Rightarrow\ \sin\theta<\frac{1}{2}$ or $\sin\theta>2$ (which is not possible)

$\therefore\ \sin\theta<\frac{1}{2}\quad\Rightarrow\theta\in\left(0,\frac{\pi}{6}\right)\cup\left(\frac{5\pi}{6},2\pi\right)$ (As $0<\theta<2\pi$)

5. A plane passes through (1, − 2, 1) and is perpendicular to two planes 2x − 2y + z = 0 and x − y + 2z = 4. The distance of the plane from the point (1, 2, 2) is

(a) 0 (b) 1 (c) 2 (d) $2\sqrt{2}$

Sol. (d)

The plane is a(x − 1) + b(y + 2) + c(z − 1) = 0

where 2a − 2b + c = 0 and a − b + 2c = 0

$\Rightarrow\frac{a}{1}=\frac{b}{1}=\frac{c}{0}$

So, the equation of plane is x + y + 1 = 0

$\therefore$ Distance of the plane from the point (1, 2, 2) = $\frac{1+2+1}{\sqrt{1^2+1^2}}=2\sqrt{2}$

6. r, s, t are prime numbers and p, q are positive integers such that their LCM is $r^2s^4t^2$, then numbers of pairs of (p, q) are

(a) 225 (b) 252 (c) 246 (d) 256

Sol. **a**

For getting LCM as $r^2s^4t^2$, one of p and q has to get 2 r's. This can be done in 5 ways. Similarly for 2 t's, number of ways = 5. For 4 s's, one of p and q has to get 4 s's. This can be done in 9 ways. So total number of pairs (p, q) = 5 × 5 × 9 = 225.

7. $\int\frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx$

(a) $\frac{\sqrt{2x^4-2x^2+1}}{x^2}$ **(b)** $\frac{\sqrt{2x^4-2x^2+1}}{x}$ **(c)** $\frac{\sqrt{2x^4-2x^2+1}}{2x^2}$ **(d)** $\frac{\sqrt{2x^4-2x^2+1}}{x^3}$

Sol. **c**

$$\int\frac{(x^2-1)x}{x^3\sqrt{2x^4-2x^2+1}}dx=\int\frac{(x^2-1)dx}{x^5\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}$$

Put $2-\frac{2}{x^2}+\frac{1}{x^4}=t$

$\Rightarrow\ \left(\frac{4}{x^3}-\frac{4}{x^5}\right)dx=dt\ \Rightarrow\ \frac{1}{4}\int\frac{dt}{\sqrt{t}}=\frac{1}{4}\times2\sqrt{t}+c=\frac{1}{2}\sqrt{2+\frac{1}{x^4}-\frac{2}{x^2}}+c$

$\Rightarrow \dfrac{1}{2x^2}\sqrt{2x^4+1-2x^2}+c$

8. $f(x)$ is a twice differentiable function such that

$f''(x)=-f(x)$

$g(x)=f'(x)$ and $F(x)=\left(f\left(\frac{x}{2}\right)\right)^2+\left(g\left(\frac{x}{2}\right)\right)^2$, $F(5)=5$, then $F(10)=$

(a) 5 (b) 10 (c) 20 (d) 0

Sol. **a**

$$F(x)=\left(f\left(\frac{x}{2}\right)\right)^2+\left(g\left(\frac{x}{2}\right)\right)^2$$

$$F'(x)=f\left(\frac{x}{2}\right).f'\left(\frac{x}{2}\right)+g\left(\frac{x}{2}\right).g'\left(\frac{x}{2}\right) = f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right)+g\left(\frac{x}{2}\right)f''\left(\frac{x}{2}\right)$$

$$= f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right)-g\left(\frac{x}{2}\right)f\left(\frac{x}{2}\right)=0$$

So $F(x)$ is a constant function, So $F(10) = 5$.

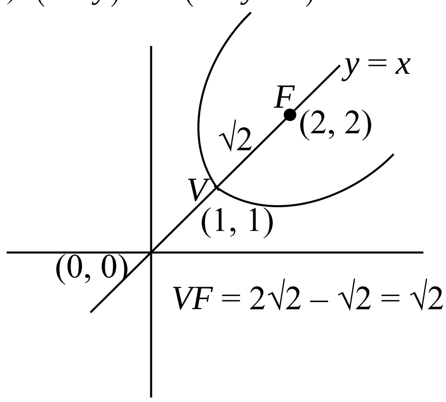
9. Axis of parabola is along the line $y = x$, its focus and vertex are at distances $2\sqrt{2}$ and $\sqrt{2}$ respectively from origin, then the equation of the parabola is (focus and vertex lie in the 1st quadrant)

(a) $(x-y)^2=8(x+y-2)$ (b) $(x+y)^2=8(x+y-2)$

(c) $(x-y)^2=4(x+y-2)$

(d) $(x+y)^2=4(x+y-2)$

Sol. **a**



Equation of directrix is

$y=-x$

i.e., $x+y=0$

Equation of parabola is

$$(x-2)^2+(y-2)^2=\left(\frac{x+y}{\sqrt{2}}\right)^2$$

$\Rightarrow$ $2(x^2-4x+4+y^2-4y+4)=x^2+y^2+2xy$

$\Rightarrow$ $x^2+y^2-2xy=8x+8y-16$

$\Rightarrow$ $(x-y)^2=8(x+y-2)$

10. If $t_1=(\tan\theta)^{\tan\theta}$, $t_2=(\tan\theta)^{\cot\theta}$, $t_3=(\cot\theta)^{\tan\theta}$, $t_4=(\cot\theta)^{\cot\theta}$ where $\theta \in \left(0,\frac{\pi}{4}\right)$, then

(a) $t_2<t_1<t_3<t_4$ (b) $t_4<t_1<t_2<t_3$

(c) $t_4<t_3<t_2<t_1$ (d) $t_3<t_4<t_2<t_1$

Sol. **a**

As $\theta \in \left(0,\frac{\pi}{4}\right)$, so $0<\tan\theta<1$, $\cot\theta>1$

$t_3=(\cot\theta)^{\tan\theta}$, $t_4=(\cot\theta)^{\cot\theta}$

As cotθ > 1, cotθ > tanθ $\Rightarrow t_4 > t_3$
Now $t_3 = (\cot\theta)^{\tan\theta}$, $t_2 = (\tan\theta)^{\cot\theta}$
As cotθ > 1 ⇒ $t_3 > 1$
As tanθ < 1, cotθ > 1 ⇒ $t_2 < 1$ ⇒ $t_3 > t_2$
Now $t_1 = (\tan\theta)^{\tan\theta}$, $t_2 = (\tan\theta)^{\cot\theta}$
As tanθ < 1 and cotθ > tanθ ⇒ $t_2 < t_1$
Now $t_1 = (\tan\theta)^{\tan\theta}$, $t_3 = (\cot\theta)^{\tan\theta}$
As cotθ > 1 ⇒ $t_3 > 1$
As tanθ < 1 ⇒ $t_1 < 1$ ⇒ $t_1 < t_3$
So correct order is $t_4 > t_3 > t_1 > t_2$.

11. If *a, b, c* are sides of Δ, $a \neq b \neq c$. $x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$. If the roots are real, then value of λ

(a) $\frac{1}{3} < \lambda < \frac{5}{3}$ **(b)** $\lambda > \frac{5}{3}$ **(c)** $\frac{4}{3} < \lambda < \frac{5}{3}$ **(d)** $\lambda < \frac{4}{3}$

Sol. d

$x^2 - 2x(a+b+c) + 3\lambda(ab+bc+ca) = 0$

As equation has real roots

⇒ **D ≥ 0**

$\Rightarrow\ 4(a+b+c)^2 - 12\lambda(ab+bc+ca) \geq 0 \Rightarrow\ 4[a^2+b^2+c^2+(ab+bc+ca)(2-3\lambda)] \geq 0$

or $\frac{a^2+b^2+c^2}{ab+bc+ca} \geq (3\lambda - 2)$

As *a*> |*b* – *c*|, *b* > |*c* – *a*|, *c* > |*a* – *b*|

$\Rightarrow\ \frac{a^2+b^2+c^2}{ab+bc+ca} < 2$ ⇒ **3λ – 2 < 2** ⇒ $\lambda < \frac{4}{3}$

12. Let $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, $\vec{c} = \hat{i} + \hat{j} - \hat{k}$. A vector in the plane of $\vec{a}$ and $\vec{b}$ whose projection on $\vec{c}$ is

$\frac{1}{\sqrt{3}}$ -

(a) $4\hat{i} - \hat{j} + 4\hat{k}$ (b) $3\hat{i} - 3\hat{j} + \hat{k}$ (c) $\hat{i} - 3\hat{j} + \hat{k}$ (d) $\hat{i} - \hat{j} + \hat{k}$

Sol. (a)

Vector lying in the plane of $\vec{a}$ and $\vec{b}$ is $\vec{r} = x\vec{a} + y\vec{b}$ and its projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$

$\Rightarrow \left[(x+y)\hat{i} + (2x-y)\hat{j} + (x+y)\hat{k}\right].\frac{(\hat{i}+\hat{j}-\hat{k})}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\Rightarrow 2x - y = 1 \Rightarrow \vec{r} = (3x-1)\hat{i} + \hat{j} + (3x-1)\hat{k}$

13 Two function are $f(x) = \begin{cases} e^x & 0 \leq x \leq 1 \\ 2 - e^{x-1} & 1 < x \leq 2 \\ x - e & 2 < x \leq 3 \end{cases}$, $g(x) = \int_0^x f(t)\,dt, x \in [1,3]$, then

(a) $g(x)$ has local minima at $x = 2$
(b) $g(x)$ has local maxima at $x = 1 + \log 2$, minima at $x = e$
(c) $g(x)$ has no local maxima
(d) $g(x)$ has no local minima

Sol. a,b

$$g'(x) = f(x) = \begin{cases} e^x & 0 \le x \le 1 \\ 2 - e^{x-1} & 1 < x \le 2 \\ x - e & 2 < x \le 3 \end{cases}$$

$g'(x) = 0$ when $x = 1 + \ln 2$ and $x = e$

$$g''(x) = \begin{Bmatrix} -e^{x-1} & 1 < x \le 2 \\ 1 & 2 < x \le 3 \end{Bmatrix}$$

$g''(1+\ln 2) = -e^{\ln 2} < 0$ hence at x =1 + ln 2, g(x) has a local maximum

$g''(e) = 1 > 0$ hence $x = e, g(x)$ has a local minimum.

Q f(x) is discontinuous at x=1, then we get local maxima at x=1 and local minima at x=2.

14 For the curve, tangent at the point P cut the coordinate axis at A and B and given PA : PB is 1 : 3 and curve passes through (1, 1), then

(a) curve passes through $\left(2, \frac{1}{8}\right)$ (b) equation of tangent at (1, 1) is $3x + y = 4$

(c) differential equation of curve is $\frac{dy}{dx} = -\frac{3y}{x}$ (d) differential equation of curve is $\frac{dy}{dx} = \frac{3y}{x}$

Sol. **a, b, c**

For the curve y = f(x) equation tangent at P(x, y)

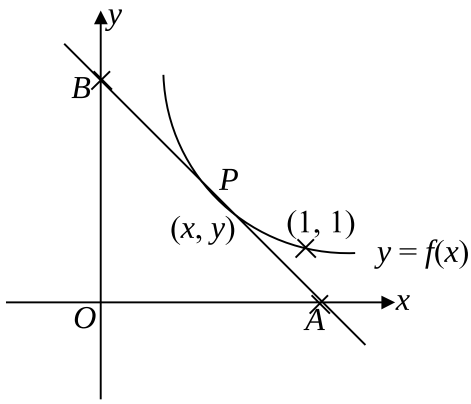
Y – y = f′(x) (X – x)

Point A ≡ $\left(x - \frac{y}{f'(x)}, 0\right)$

$B \equiv (0, y - xf'(x))$

$\frac{PA}{PB} = \frac{3}{1}$ using section formula

$$x = \frac{3\left(x - \frac{y}{f'(x)}\right)}{4}$$



$\Rightarrow \frac{dy}{dx} = -\frac{3y}{x}$ $\Rightarrow y = \frac{c}{x^3}$

curve sPasses through (1, 1) , So $y = \frac{1}{x^3}$

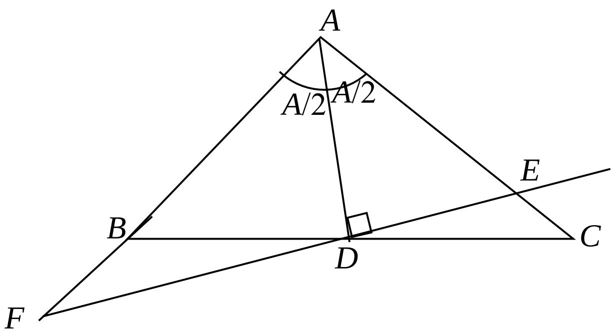
15 AD is the angular bisector of ∠A. DE ⊥ AD intersects AC at E and meets AB produced at F, then

(a) ΔAEF is isosceles (b) AD = $\frac{2bc}{b+c}\cos\frac{A}{2}$

(c) AE is harmonic mean of b and c (d) AF = $\frac{4bc}{b+c}\sin\frac{A}{2}$

Sol. **a, b, c**

Obviously AD = $\frac{2bc}{b+c}\cos\frac{A}{2}$



$\Delta AED \approx \Delta AFD$ $\Rightarrow$ AE = AF

So ΔAEF is isosceles

In ΔAED, $\cos\frac{A}{2} = \frac{AD}{AE} \Rightarrow AE = \frac{2bc}{b+c}$.

16. Equations of common tangents to the parabola $x^2 = y$ and $y = -x^2 + 4x - 4$

(a) y = 0 (b) y = 4(x – 1)

(c) y = 4(x + 1) (d) y = –4(x – 1)

sol. a,b

Any tangent on $x^2 = y$, having slope *m* can be taken as $y = mx - \frac{m^2}{4}$.

As this is also tangent to $y = -x^2 + 4x - 4$

$\Rightarrow$ *D* will be zero of the equation $\Rightarrow mx - \frac{m^2}{4} = -x^2 + 4x - 4$

i.e. $x^2 + x(m-4) + \left(4 - \frac{m^2}{4}\right) = 0$ $\Rightarrow (m-4)^2 - 4 \times 1\left(4 - \frac{m^2}{4}\right) = 0$

$m^2 + 16 - 8m = 16 - m^2$ $\Rightarrow 2m^2 - 8m = 0 \Rightarrow 2m(m-4) = 0$

$m = 0 \quad m = 4$

$\therefore$ Equations of common tangents are

$y = 0, y = 4x - 4 = 4(x - 1)$

17 $f(x) = \min(1, x^2, x^3)$, then

(a) $f'(x) > 0$, $\forall x > 1$ (b) $f(x)$ is always continuous

(c) $f(x)$ is continuous but not differentiable $\forall x \in R$ (d) $f(x)$ is not differentiable at two points

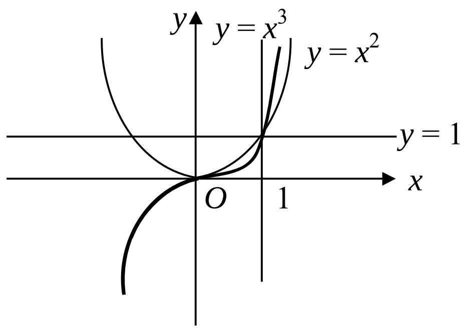
Sol. **b, c**

As $f(x) = \min\{1, x^2, x^3\}$

So $f(x) = \begin{cases} x^3 & x \le 1 \\ 1 & x > 1 \end{cases}$

So function is continuous $\forall$ $x \in R$ but not differentiable at $x = 1$

Also $f'(0) = 0$ $\forall x > 1$



18 A hyperbola is such that it passes through the focus of ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and product of their eccentricities is 1. Also transverse and conjugate axis of hyperbola coincides with major and minor axis of the ellipse respectively, then

(a) equation of hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$ (b) equation of hyperbola $\frac{x^2}{9}-\frac{y^2}{25}=1$

(c) focus of hyperbola (5, 0) (d) focus of hyperbola is $(5\sqrt{3},\, 0)$

Sol. **a, c**

For ellipse, $a = 5$, $b = 4$

$e^2 = 1-\frac{16}{25}=\frac{9}{25}$ $e = 3/5$

So $ae = 3$

So foci of ellipse are (3, 0) and (–3, 0)

Eccentricity of hyperbola = 5/3

So the hyperbola satisfies above conditions.

For this hyperbola, $a = 3$, $e = 5/3$ s$\Rightarrow ae = 5$

So focus of hyperbola is (5, 0).

19. f(x) is cubic polynomial which has local maximum at x = − 1. If f(2) = 18, f(1) = −1 and f′(x) has local minima at x = 0, then

(a) the distance between (−1, 2) and (a, f(a)), where x = a is the point of local minima is $2\sqrt{5}$

(b) f(x) is increasing for $x\in\left[1,2\sqrt{5}\right]$

(c) f(x) has local minima at x = 1

(d) the value of f(0) =5

Sol. **(b), (c)**

Let $f(x)=ax^3+bx^2+cx+d$

$f'(x)=3ax^2+2bx+c$

$f''(x)=6ax+2b$

$f'(x)$ has minimum at $x = 0$ $\Rightarrow$ $f''(0)=0 \Rightarrow b=0$ (1)

$f(x)$ has maximum at $x=-1$ $\Rightarrow f'(-1)=3a-2b+c=0 \Rightarrow 3a=-c$ (2)

$f(-1)=-a+b-c+d=2$ (3)

$f(3)=27a+9b+3c+d=18$ (4)

Solving equation (1), (2), (3) and (4), we get

$\Rightarrow$ $a = 1, b = 0, c = -3, d = 0.$ $\Rightarrow$ $f(x)=x^3-3x$

$f'(x)=3x^2-3$

$f'(x)=0 \Rightarrow x=\pm 1$

$f''(x)=6x$

$f''(1)=6>0$ $\Rightarrow x = 1$ is local minima.

The required polynomial which satisfy the condition

is f(x) = ¼ ($19x^3-57x+34$)

f(x) has local maximum at x = −1 and local minimum at x = 1

Hence f(x) is increasing for $x\in\left[1,2\sqrt{5}\right]$

20. Let $\bar{A}$ be vector parallel to line of intersection of planes P_1 and P_2 through origin P_3 is parallel to the vectors $2\hat{j}+3\hat{k}$ and $4\hat{j}-3\hat{k}$ and is parallel to $\hat{j}-\hat{k}$ and $3\hat{i}+3\hat{j}$, then the angle between vectors $\bar{A}$ and $2\hat{i}+\hat{j}-2\hat{k}$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{3\pi}{4}$

Sol. (b), (d)

Vector AB is parallel to $\left[\left(2\hat{i}+3\hat{k}\right)\times(4)-3\hat{k}\right]\times\left[\left(\hat{j}-\hat{k}\right)\times\left(3\hat{i}+3\hat{j}\right)\right]=54\left(\hat{j}-\hat{k}\right)$

Let θ is the angle between the vector, then

$\cos\theta=\pm\left(\frac{54+108}{3.54\sqrt{2}}\right)=\pm\frac{1}{\sqrt{2}}$

Hence $\theta=\frac{\pi}{4},\frac{3\pi}{4}$

Passage I

$\int_a^b f(x)dx$ represents the area bounded between curve $y = f(x)$ and x–axis and ordinate $x = a$, $x = b$,

then $\int_a^b f(x)dx=\frac{b-a}{2}\left[f(a)+f(b)\right]$. If $c \in (a, b)$ then $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$,

$\int_a^b f(x)dx=\frac{c-a}{2}\left(f(a)+f(c)\right)+\frac{b-c}{2}\left(f(b)+f(c)\right)=F(c)$.

If $c=\frac{a+b}{2}$

Then $\int_a^b f(x)dx=\frac{b-a}{4}\left[f(a)+2f(c)+f(b)\right]$ (1)

Then answer the following questions

21 By using equation (1), the value of $\int_0^{\pi/2}\sin x dx$ is

(a) $\frac{\pi}{8}\left(1+\sqrt{2}\right)$ (b) $\frac{\pi}{4\sqrt{2}}\left(1+\sqrt{2}\right)$

(c) $\frac{\pi}{4}\left(1+\sqrt{2}\right)$ (d) $\frac{\pi}{4}\left(1+\sqrt{2}\right)$

Sol. Let $f(x)=\sin x$

$$\int_0^{\pi/2} f(x)dx=\frac{\left(\frac{\pi}{2}-0\right)}{4}\left[f\left(\frac{\pi}{2}\right)+2.f\left(\frac{\pi}{4}\right)+f(0)\right]$$

$$=\frac{\pi}{8}\left[1+2.\frac{1}{\sqrt{2}}+0\right]$$

$$=\frac{\pi}{8}\left(1+\sqrt{2}\right)$$

22 $\lim_{t\to a}\frac{\int_a^t f(x)dx-\frac{t-a}{2}\left(f(t)+f(a)\right)}{(t-a)^3}=0$, then the atmost value of degree of function $f(x)$ is

(a) 4 (b) 3
(c) 2 (d) 1

Sol. $\lim_{t\to a}\dfrac{\int_a^t f(x)dx-\frac{(t-a)}{2}(f(t)+f(a))}{(t-a)^3}.\left(\frac{0}{0}\right)$

Diff. with respect to *t*, we get,

$$=\lim_{t\to a}\frac{\frac{1}{2}f(t)-\frac{1}{2}f(a)-\frac{1}{2}(t-a)f'(t)}{3(t-a)^2}\left(\frac{0}{0}\right)$$

Diff. with respect to *t*, we get,

$$\lim_{t\to a}\frac{-\frac{t-a}{2}f''(t)}{6(t-a)}=-\frac{1}{12}f''(a)=0$$

So atmost degree of function $f(x)$ is 1.

23 If $f''(x)<0$, $x\in(a, b)$ $(c, f(c))$ is a point on curve $y=f(x)$ such that $f(c)$ is maximum, then $f'(c)=$

(a) $\dfrac{f(b)-f(a)}{b-a}$ (b) 0

(c) $\dfrac{2(f(b)-f(a))}{b-a}$ (d) $\dfrac{(f(b)-f(a))^2}{b-a}$

Sol. Since, curve is concave downward and $f(c)$ is maximum, so $f'(c) = 0$

Passage II

There are n urns numbered 1 to n. Each urn contains $(n + 1)$ balls. i^{th} numbered urn contains i white balls and $(n + 1 - i)$ red balls. E is the event of selecting even numbered urn, w is the event of drawing white ball. u_i is the event that i^{th} numbered urn is selected.

24. **I**f $P(u_i)\propto i$ and one urn is selected at random, then P(w) is

(a) $\dfrac{n}{n+1}$ (b) $\dfrac{n}{2(n+1)}$

(c) $\dfrac{2n+1}{3(n+1)}$ (d) $\dfrac{1}{n+1}$

Sol. c

$P(u_i)\propto i$

$\Rightarrow P(u_i)=k\,i$

Now $P(u_1)+P(u_2)+.........+P(u_n)=1$

$\Rightarrow k+2k+\ldots..+nk=1$

$\Rightarrow k=\dfrac{2}{n(n+1)}$

$P(w)=P(w\cap u_1)+P(w\cap u_2)+\ldots+P(w\cap u_n)$

$=\sum_{i=1}^{n}\dfrac{2i}{n(n+1)}.\dfrac{i}{n+1}$

$$= \frac{2}{n(n+1)^2}\frac{n(n+1)(2n+1)}{6} = \frac{2n+1}{3(n+1)}.$$

25. If $P(u_i) = \frac{1}{n}$ and one urn is selected at random, then P(w) is

(a) $\frac{1}{2}$ (b) n

(c) $\frac{n}{n+1}$ (d) $\frac{n+1}{n+2}$

Sol. a

$$P(w) = \sum_{i=1}^{n} P(w \cap u_i)$$

$$= \sum_{i=1}^{n} \frac{1}{n} \times \frac{i}{n+1} = \frac{1}{n(n+1)}\frac{n(n+1)}{2} = \frac{1}{2}$$

26. If P(E) = c, $P\left(\frac{w}{E}\right)$ is n is an even number, then $p\left(\frac{w}{E}\right)$ is

(a) $\frac{1}{2}$ (b) $\frac{n}{n+1}$

(c) $\frac{n+2}{2(n+1)}$ (d) $\frac{n+1}{n+2}$

Sol. c

$$P\left(\frac{w}{E}\right) = \frac{P(w \cap E)}{P(E)}$$

$$= \frac{c\left[\frac{2}{n+1} + \frac{4}{n+1} + \frac{6}{n+1} + ... + \frac{n}{n+1}\right]}{c + c + c + ...\frac{n}{2} terms}$$

$$= \frac{n+2}{2(n+1)}$$

Passage III

ABCD is a square C_1 is a circle inscribed in this square and C_2 is a circle passing through the vertices of square. There is a variable point S which is equidistant from a fixed line L and a fixed point R, circle C touches the locus of point S and C_1 externally (Let P and Q be any point on the circle C_1 and C_2 respectively)

27. The value of $\frac{PA^2 + PB^2 + PC^2 + PD^2}{QA^2 + QB^2 + QC^2 + QD^2}$ is

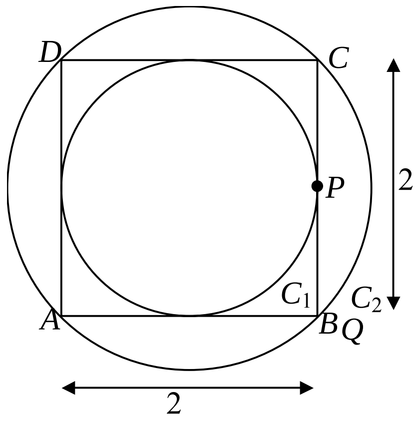
(a) 3/2 (b) 3/4

(c) 1 (d) 9

Sol. Point P lies on circle C_1
Point Q lies on circle C_2
Let us take a particular points on C_1 and C_2 as shown in figure (i.e., P as mid point of B and C and Q as B)

$$\frac{PA^2+PB^2+PC^2+PD^2}{QA^2+QB^2+QC^2+QD^2}$$

$$=\frac{12}{16}=\frac{3}{4}=0.75$$

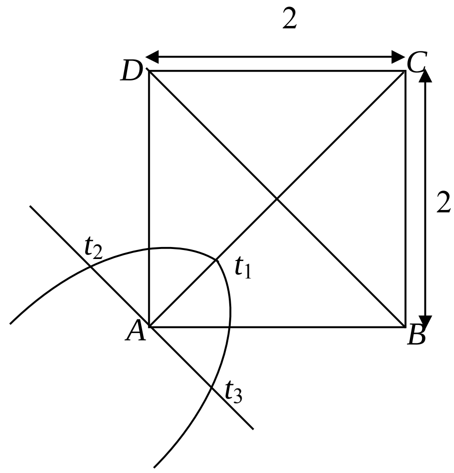


28. If fixed point is A and line L passes through points B and D adjacent to point A. Line joining AC intersect the locus of point S at point t_1. A line through A parallel to line L intersects the locus at point t_2 and t_3. The area of triangle joining t_1, t_2, t_3.

(a) 4 (b) 2
(c) 1 (d) 3

Sol. Area of triangle $=\frac{1}{2}\left(\frac{AC}{4}\right)$ (length of latus rectum)

$$=\frac{1}{2}\left(\frac{AC}{4}\right)(AC)=1.$$



29. A circle touch the line L and the circle C1 externally such that both the circles are on the same side of the line, then the locus of centre of the circle is

(a) ellipse (b) hyperbola (c) parabola (d) parts of straight line

Sol. (c)

Passage IV

If matrix $A=\begin{bmatrix}1&0&0\\2&1&0\\3&2&1\end{bmatrix}$, $\vec{u}_1,\vec{u}_2$ and $\vec{u}_3$ are three column vectors. Such that

$$A\vec{u}_1=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad A\vec{u}_2=\begin{bmatrix}2\\3\\0\end{bmatrix},\quad A\vec{u}_3=\begin{bmatrix}3\\2\\1\end{bmatrix}$$

u is a 3x3 matrix whose columns are $\vec{u}_1,\vec{u}_2$ and $\vec{u}_3$, then

30. $|u| =$

(a) 1
(b) 2
(c) 3
(d) 4

Sol. **c**

Let $u_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$

$Au_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$2x_1 + z_1 = 1$

$3y_1 + 2z_1 = 0$

$x_1 + z_1 = 0$

$\Rightarrow\ x_1 = 1,\ y_1 = \frac{2}{3},\ z_1 = -1$ $\quad\therefore\ u_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Similarly $u_2 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}$ and $u_3 = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$ $\quad\Rightarrow\ u = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & -4 \\ 1 & -4 & 0 \end{bmatrix}$

31. Sum of elements of u^{-1} is

(a) –1/3
(b) 0
(c) –7
(d) –1

Sol. c

$$u = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & -4 \\ 1 & -4 & 0 \end{bmatrix}$$

$$u^{-1} = \frac{1}{3}\begin{bmatrix} -16 & -12 & -5 \\ -4 & -3 & 2 \\ 9 & 6 & 3 \end{bmatrix}$$

Sum of elements = -7

32. $\begin{bmatrix} 3 & 2 & 0 \end{bmatrix} u \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} =$

(a) 1
(b) 3
(c) 4
(d) 5

Sol. **d**

$$\begin{bmatrix}3 & 2 & 0\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ -2 & -1 & -4\\ 1 & -4 & 0\end{bmatrix}\begin{bmatrix}3\\2\\0\end{bmatrix}=5$$

33. Find the value of $\dfrac{5050\int_0^1 \left(1-x^{50}\right)^{100} dx}{\int_0^1 \left(1-x^{50}\right)^{101} dx}$

Sol. $$\frac{5050\int_0^1 \left(1-x^{50}\right)^{100} dx}{\int_0^1 x^{5050}\left(\frac{1}{x^{50}}-1\right)^{101} dx}$$

$$\frac{5050\int_0^1 \left(1-x^{50}\right)^{100} dx}{\left.\frac{x^{5051}}{5051}\left(\frac{1}{x^{50}}-1\right)^{101}\right|_0^1 - \int_0^1 \frac{x^{5051}}{5051}\times 101\times\left(\frac{1}{x^{50}}-1\right)^{100}\times\frac{-50}{x^{51}}dx}$$

$$=\frac{5050\int_0^1 \left(1-x^{50}\right)^{100} dx}{0+\frac{5050}{5051}\int_0^1 \left(1-x^{50}\right)^{100} dx}=5051.$$

34. $A_n=\left(\frac{3}{4}\right)-\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+......+\left(-1\right)^{n-1}\left(\frac{3}{4}\right)^n$, $B_n=1-A_n$. Find a least natural number n_0, so that $B_n > A_n$. $\forall\ n \geq n_0$.

Sol. $B_n=1-A_n>A_n$

$\Rightarrow\ A_n<\frac{1}{2}$

Now $A_n=\dfrac{\frac{3}{4}\left(1-\left(-\frac{3}{4}\right)^n\right)}{1+\frac{3}{4}}<\dfrac{1}{2}$ $\qquad\Rightarrow\ \left(-\frac{3}{4}\right)^n>-\frac{1}{6}$

Least natural number n satisfying above condition is 6.

35. If $f(x)$ is twice differentiable function such that

$f(a)=0,\ f(b)=2,\ f(c)=-1,\ f(d)=2,\ f(e)=0$

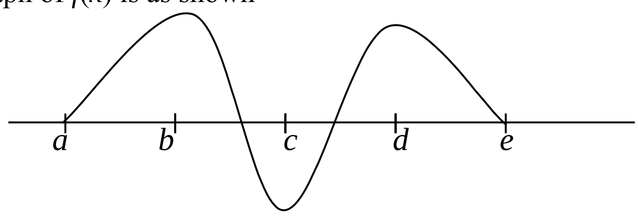
If $g(x)=\left(f'(x)\right)^2+f(x).f''(x)$

Then find the minimum number of roots of $g(x)=0$ in $[a, e]$

Sol. $g(x) = (f(x) f'(x))'$

Let $h(x) = f(x) f'(x)$, then $g(x) = h'(x)$

Graph of $f(x)$ is as shown



According to above graph , f(x) will be zero at 4 points and by Rolle's theorem $f'(x)$ will be zero for atleast three values of x.

Hence $h'(x)$ will be zero for atleast 7 values of x.

Now as there will lie at least one root of $g(x) = 0$ inbetween points where $g(x)$ has opposite signs,So, minimum number of roots of $g(x) = 7$

36. a, b, c and d are real and distinct number a and b are roots of equation $x^2 - 10cx - 11d = 0$ and c and d are roots of $x^2 - 10ax - 11b = 0$, then find the value of $a + b + c + d$.

Sol. $a + b = 10c$

$ab = -11d$

$c + d = 10a$

$cd = -11b$

$a + b + c + d = 10(a + c)$ (A)

As $a + b = 10c$ $\qquad$ $c + d = 10a$

Let $a = 5c - \alpha$ (i) $\qquad$ $c = 5a - \beta$ (ii)

$b = 5c + \alpha$ $\qquad$ $d = 5a + \beta$

As $ab = -11d \Rightarrow 25c^2 - \alpha^2 = -11d$

$25c^2 - \alpha^2 = -11(5a + \beta)$ (1)

As $cd = -11b \Rightarrow 25a^2 - \beta^2 = -11b$

$25a^2 - \beta^2 = -11(5c + \alpha)$ (2)

(1) – (2) gives

$$25(c^2 - a^2) + (\beta^2 - \alpha^2) = -11(5a + \beta - 5c - \alpha)$$

$$\Rightarrow \quad 25(c-a)(c+a) - 55(c-a) = (\beta - \alpha)[(-11) - (\beta + \alpha)]$$

(i) – (ii) $\Rightarrow$ $5(a-c) + (\alpha - \beta) = c - a$

$$6(a-c) = (\beta - \alpha)$$

$$\Rightarrow \quad (c-a)[25(c+a) - 55] = 6(a-c)[-11 - \beta - \alpha]$$

$$\Rightarrow \quad 55 - 25(c+a) = -6(11 + \alpha + \beta)$$

(i) – (ii)$\Rightarrow$ $a + c = 5(a+c) - (\alpha + \beta) \Rightarrow \alpha + \beta = 4(a + c)$

$$(a+c) = 55 + 66 = 121$$

$\Rightarrow$ $a + b + c + d = 1210$.

Match the following

37. Normals at 3 points P, Q, R to the parabola $y^2 = 4x$ intersect at (3, 0). Then match the following

I.	Area of ΔPQR	(a)	2
II.	Centroid of ΔPQR	(b)	5/2
III.	Circumcentre of ΔPQR	(c)	$\left(\frac{2}{3}, 0\right)$

IV. Circumradius of ΔPQR (d) $\left(\frac{5}{2}, 0\right)$

Sol. I –(a) , II – © , III- (d) , IV-(b)

Equation of normal to $y^2 = 4x$ is

$$y = -tx + 2t + t^3$$

It passes through(3, 0)

$\Rightarrow$ $t^3 - t = 0$

$\Rightarrow$ $t = 0, \pm 1$

So conormal points are (0, 0), (1, 2) and (1, –2)

So centroid = $\left(\frac{2}{3}, 0\right)$

Area of $\Delta PQR = 2$

Circumcentre = $\left(\frac{5}{2}, 0\right)$ and circumradius = $\frac{5}{2}$

38. Match the following

I.	$\int_0^{\pi/2} (\sin x)^{\cos x} \left(\cos x \cot x - \log(\sin x)^{\sin x}\right) dx$	(a)	1
II.	Area bounded by $-4y^2 = x$ and $x - 1 = -5y^2$	(b)	0
III.	Cosine of the angle of intersection of curves $y = 3^{x-1}\log x$ and $y = x^x - 1$ is	(c)	6 ln 2
IV.	Data could not be retrieved.	(d)	4/3

Sol. **I – (a), II – (d), III – (b)**

(i) $I = \int_0^{\pi/2} (\sin x)^{\cos x} (\cos x.\cot x - \log(\sin x)^{\sin x}) dx$

$\Rightarrow I = \int_0^{\pi/2} \frac{d}{dx} (\sin x)^{\cos x} dx = 1$

(ii). The points intersection of $-4y^2 = x$ and $x - 1 = -5y^2$ is (-4,1)

Hence required area $= 2\left|\left[\int_0^1 (1 - 5y^2) dy - \int_0^1 -4y^2 dy\right]\right| = \frac{4}{3}$

(iii) The point of intersection of $y = 3^{x-1}\log x$ and $y = x^x - 1$ is (1,0)

Hence $\frac{dy}{dx} = \frac{3^{x-1}}{x} + 3^{x-1}\log 3 \log x. \left.\frac{dy}{dx}\right|_{(1,0)} = 1$

for $y = x^x - 1. \left.\frac{dy}{dx}\right|_{(1,0)} = 1$

If θ is the angle between the curve then $= \theta \Rightarrow \cos\theta = 1$

39. Match the following

I.	Two rays in the first quadrant x + y = |a| and ax = y - 1 intersects each other in the interval , the value of is	(a)	2
II.	Point (α,β,γ) lies on the plane x + y + z = 2. Let $\vec{a} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}, \hat{k}\times(\hat{k}\times a) = 0,$ they	(b)	4/3
III.	$\left\|\int_0^1 (1-y^2)dy\right\| + \left\|\int_1^0 (y^3-1)dy\right\|$	(c)	$\left\|\int_0^1 \sqrt{1-x}dx\right\| + \left\|\int_1^0 \sqrt{1+x}dx\right\|$
IV.	If $\sin A\sin B\sin C + \cos A\cos B = 1$, then the value of sinC=	(d)	1

.

Sol. **I – (d), II – (a), III – (b,c), IV – (d)**

(i) Solving the two equations of ray i.e. x + y = |a| and ax - y = 1

we get $x = \frac{|a|+1}{a+1} > 0$ and $y = \frac{|a|-1}{a+1} > 0$

when a + 1 > 0, we get a > 1 $\therefore\ a_0 = 1$

(ii) We have $\vec{a} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k} \Rightarrow \vec{a}.\hat{k} = \gamma$

Now, $\hat{k}\times(\hat{k}\times\hat{a}) = (\hat{k}.\vec{a})\hat{k} - (\hat{k}.\hat{k})\vec{a}$

$= \gamma\hat{k} - (\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k})$

$= \alpha\hat{i} + \beta\hat{j} = \vec{0} \Rightarrow \alpha - \beta = 0$

As $= \alpha + \beta + \gamma = 2 \Rightarrow \gamma = 2$

(iii) $\left|\int_0^1 (1-y^2)dy\right| + \left|\int_0^1 (y^2-1)dy\right| = 2\int_1^1 (1-y^2)dy = \frac{4}{3}$

$= \left|\int_0^1 \sqrt{1-x}dx\right| + = \left|\int_1^0 \sqrt{1+x}dx\right| = 2\int_0^1 \sqrt{1-x}dx$

$= 2\int_0^1 \sqrt{x}dx = 2.\frac{2}{3}.x^{3/2}\Big|_0^1 = \frac{4}{3}$

(iv) $\sin A\sin B\sin C + \cos A\cos B \le \sin A\sin B + \cos A\cos B = \cos(A-B)$

$\Rightarrow \cos(A-B) \ge 1 \Rightarrow \cos(A-B) = 1 \Rightarrow \sin C = 1$

40. Match the following

I.	$\sum_{i=1}^{x} \tan^{-1}\left(\frac{1}{2i^2}\right) = t$, then tan t =	(a)	2/3
II.	Sides a,b,c of a triangle ABC are in AP	(b)	2/9

and

$\cos\theta_1 = \frac{a}{b+c}, \cos\theta_2 = \frac{b}{a+c}, \cos\theta_3 = \frac{c}{a+b},$

then $\tan^2\left(\frac{\theta_1}{2}\right) + \tan^2\left(\frac{\theta_3}{2}\right) =$

III. A line is perpendicular to x + 2y + 2z = 0 and passes through The perpendicular distance of this line from the origin is (c) $\pi/4$

IV. Data could not be retrived. (d)

.

Sol. **I. – (c) , II. – (a), III. – (b)**

$\sum_{i=1}^{\infty} \tan^{-1}[\frac{1}{2i^2}] = t$ now, $\sum_{i=1}^{\infty} \tan^{-1}[\frac{2}{4i^2 - 1 + 1}]$

$\sum_{i=1}^{\infty} [\tan^{-1}(2i+1) - \tan^{-1}(2i-1)]$

$[(\tan^{-1}3 - \tan^{-1}1) + (\tan^{-1}5 - \tan^{-1}3) + ... + \tan^{-1}(2n+1) - \tan^{-1}(2n-1)..........\infty]$

$t = \tan^{-1}(2n+1) - \tan^{-1} = \tan^{-1}\frac{2n}{1+(2n+1)}$ $\Rightarrow \tan t = \frac{n}{n+1} \Rightarrow t = \frac{\pi}{4}$

(ii) We have $\Rightarrow \tan^2\frac{\theta_1}{2} = \frac{b+c-a}{b+c+a}$

Also $\cos\theta_3 = \frac{1-\tan^2\frac{\theta_1}{2}}{1+\tan^2\frac{\theta_1}{2}} = \frac{c}{a+b} = \tan^2\frac{\theta_3}{2} = \frac{a+b+c}{a+b+c}$

$\therefore \tan^2\frac{\theta_1}{2} + \tan^2\frac{\theta_3}{2} = \frac{2b}{3b} = \frac{2}{3}$

(iii) Line through $(0,1,0)$ and perpendicular to plane $x+2y+2z=0$ is given by $\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-1}{2} = r$

Let $P(r, 2r+1, 2r)$ be the perpendicular on the straight line then

$r \times 1 + (2r+1) + 2 \times 2r = 0 \Rightarrow r = \frac{2}{9}$